Föreläsning 27/11-13

6.9 Continous time Markov chains

Definition: A continous time discrete valued stochastic process $\{X(t); t \geq 0\}$ is Markov if $P(X(t_n)=j \mid X(t_{n-1})=i_{n-1}, \dots, X(t_1)=i_1) = P(X(t_n)=j \mid X(t_{n-1})=i_{n-1})$ for $t_1 < \dots < t_{n-1} < t_n$

Transition probability $P_{ij}(t) = P(X(t+s)=j \mid X(s)=i)$ is assumed not to depend on s = time homogenity.

Transition matrix $P_t = (P_{ij}(t))_{ij}$

Probability distribution row matrix $\mu^{(t)}$ of $X(t)$ with elements $\mu^{(t)}_i = P(X(t)=i)$

Chapman Kolmogorov $P_{s+t} = P_s P_t \qquad \mu^{(t)} = \mu^{(0)} P_t$

Generator $G = \lim\limits_{h\to 0} \frac{P_h - I}{h} = P_0' \qquad (I = P_0)$

Forward equation $P_t' = P_t G$

Backward equation $P_t' = G P_t$

Thm

$$P_t = e^{tG} = \sum_{n=0}^{\infty} \frac{(tG)^n}{n!}$$

Proof of forward equation:

$$\text{r.h} = P_t G = \lim_{h\to 0} \frac{P_t P_h - P_t I}{h} = \lim_{h \to 0} \frac{P_{t+h} - P_t}{h} = P_t' = \text{l.h} \quad \blacksquare$$

Proof of backward equation:

Entirely similar ... $\blacksquare$

Proof of thm:

Same as in univariate basic stochastic,

$$\frac{d}{dt}(e^{tG}) = \sum_{n=1}^{\infty} n \frac{t^{n-1} G^n}{n!} = G \underbrace{\sum_{n=1}^{\infty} \frac{(tG)^{n-1}}{(n-1)!}}_{e^{tG}}$$

$$\Rightarrow \frac{d}{dt}(e^{tG}) = G e^{tG} \quad \blacksquare$$

Next week in excercise (6.9.1)

$$G = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix} \quad B = \begin{pmatrix} 1 & \mu \\ 1 & -\lambda \end{pmatrix} \quad G = B \begin{pmatrix} 0 & 0 \\ 0 & -(\lambda+\mu) \end{pmatrix} B^{-1}$$

$$e^{tG} = B \begin{pmatrix} 1 & 0 \\ 0 & e^{-(\lambda+\mu)\lambda} \end{pmatrix} B^{-1} = P_t$$

Discrete time Markov understood by looking at P.

Continous time Markov understood by looking at G.

Basic facts for generator:

$G = P_0'$ means that $P_{ij}(h) = \begin{cases} g_{ij}h + o(h) & i \neq j \\ 1 + g_{ii}h + o(h) & i = j \end{cases}$ as $h \to 0$

$$P_{ij}(h) = \underbrace{P_{ij}(0)}_{\delta_{ij}} + P_{ij}'(0)h + o(h)$$

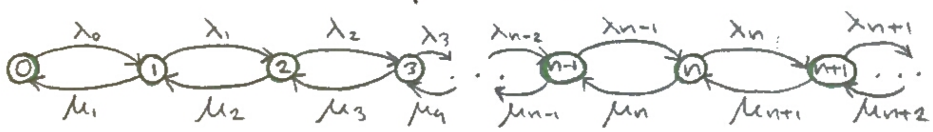
$$1 = \sum_j P_{ij}(h) = \sum_{j \neq i} g_{ij}h + o(h) + 1 + g_{ii}h + o(h) \Rightarrow \sum_j g_{ij} = 0$$

$$0 \leq -g_{ii} = \sum_{j \neq i} g_{ij}$$

Main fact

A continous time Markov chain spends an $\exp(-g_{ii})$ distributed time in state i. After that it moves to a new state j where the probability for different j are: $\frac{g_{ij}}{\sum_{j \neq i} g_{ij}} = \frac{g_{ij}}{-g_{ii}}$ for $j \neq i$.

6.11 Birth and death processes



$$G = \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array}$$

Spend $\exp(\lambda_n+\mu_n)$-dist. time in n.

$\begin{cases} \text{one step up from } n \text{ up} & \lambda_n/(\lambda_n+\mu_n) \\ \text{— " — down — " —} & \mu_n/(\lambda_n+\mu_n) \end{cases}$

Birth and death process

Birth process = birth and death process with no deaths, i.e., all $\mu_n = 0$.

$$0 \xrightarrow{\lambda_0} 1 \xrightarrow{\lambda_1} 2 \xrightarrow{\lambda_2} 3 \xrightarrow{\lambda_3} 4 \xrightarrow{\lambda_4} 5 \xrightarrow{\lambda_5} \cdots$$

Spends $\exp(\lambda_n)$-dist. time in n, then moves to $n+1$

Poisson process = birth process with $\lambda_n = \lambda \quad \forall n$.

$$0 \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} 3 \xrightarrow{\lambda} 4 \xrightarrow{\lambda} 5 \xrightarrow{\lambda} \cdots$$

Spends $\exp(\lambda)$-dist. time in n, then moves to $n+1$.

π is stationary distribution if $\pi P_t = \pi$ for all $t \geq 0$, then $\mu^{(0)} = \pi \Longrightarrow \mu^{(t)} = \pi$ all $t \geq 0$.

__Thm__

π is stationary dist. $\Longleftrightarrow \pi G = 0$

__Proof__: $\boxed{\Longleftarrow}$ Assume $\pi G = 0$, then $\pi P_t = \pi e^{tG} = \pi \sum_{n=0}^{\infty} \frac{(tG)^n}{n!} =$

$= \pi + \sum_{n=1}^{\infty} \underbrace{\pi G}_{=0} \frac{t^n G^{n-1}}{n!} = \pi$ ▨ ($\boxed{\Longrightarrow}$ similar...)

Definition: Markov chain is irreducible if $p_{ij}(t) > 0$ for some t for every choice of ij. Then M.C. is irreducible iff $p_{ij}(t) > 0$ for all t for every choice of i, j.

__Proof sketch__: we skip it...

__Main thm__

For irreducible chain $p_{ij}(t) \to \pi_j$ as $t \to \infty \quad \forall j, i$ (& π exists).
If π do not exist then $p_{ij}(t) \to 0$ as $t \to \infty \quad \forall i, j$.

Take a look at Poisson process (in sec. 6.8)
Take a look at Forward eq. for ditto.

$$P_t' = P_t G \Longleftrightarrow \begin{cases} P_0'(t) = -\lambda P_0(t) \\ P_n'(t) = \lambda P_{n-1}(t) - \lambda P_n(t) & n \geq 1 \end{cases} \qquad G = \begin{pmatrix} -\lambda & \lambda & & 0 \\ & -\lambda & \lambda & \\ 0 & & & \ddots \end{pmatrix}$$

where $P_n(t) = P(X(t) = n) = \mu_n^{(t)} = (\mu^{(0)} P_t)_n = \{\mu^{(0)} = (1\ 0\ 0 \ldots)\} =$

$= (\text{first row of } P_t)_n$

Differentiate this: first row of $P_t' =$ first row of $P_t G$

$$\begin{cases} p_{00}'(t) = -\lambda p_{00}(t) \\ p_{0n}'(t) = \lambda p_{0n-1}(t) - \lambda p_{0n}(t) \end{cases}$$

Solve this by induction to get $P_n(t) = p_{0n}(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$